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On the stability of electron plasma waves

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Abstract. Electron plasma waves in both isothermal and adibatic electron gases are shown to be stable with respect to small amplitude, three-dimensional perturbations.

1. Introduction

Nonlinear plasma waves in a cold electron plasma are by now well known and have been shown to be stable (see Davidson 1971, Infeld 1972, Infeld and Rowlands 1973, Gribben and Parkes 1977). The same kind of waves in a warm plasma, however, never seem to have been investigated for stability. Indeed there even seem to be some rather dubious statements about their very nature in the literature. It has been suggested, for example, that electron plasma *solitons* can exist, and also that approximations are needed before the form of the nonlinear waves can be obtained from the governing equations. Both these statements are, in our opinion, incorrect and, together with the relevant references, will be dealt with in the Appendix.

The plasma waves which we will investigate for stability are the nonlinear generalisations of the electron plasma modes with the dispersion relation

$$\omega^2 = \omega_p^2 + \text{const.} \times k^2. \tag{1.1}$$

When nonlinear waves exist they propagate with arbitrary phase velocity and thus differ fundamentally from equation (1.1). In our analysis we will show the existence of nonlinear waves in adiabatic and isothermal plasmas and study their stability to three-dimensional perturbations.

2. Form of the nonlinear wave

The fluid equations governing an electron gas are

$$\begin{aligned} &\partial n/\partial t + (\partial/\partial \boldsymbol{x})(n\boldsymbol{v}) = 0 \\ &\partial \boldsymbol{v}/\partial t + \boldsymbol{v} \,.\, \partial \boldsymbol{v}/\partial \boldsymbol{x} = \nabla\phi - (1/n)\nabla p \\ &\nabla^2\phi = n - 1 \\ &(\mathrm{D}/\mathrm{D}t)(p/n^\gamma) = 0 \qquad \text{(adiabatic)} \\ &p = n \qquad \text{(isothermal).} \end{aligned} \tag{2.1}$$

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Here the variables have been 'nondimensionalised'. In the isothermal case the electron temperature T is constant, v is normalised by $(T/m_e)^{1/2}$, t by ω_p^{-1}, ϕ by T/e, x by λ_D, and p by $n_0T(\omega_p^2 = 4\pi n_0 e^2/m_e, \lambda_D^2 = T/4\pi n_0 e^2)$. For the adiabatic case p_0/n_0 should be substituted for T in the above scheme. Boltzmann's constant has been taken to be unity and n, normalised to n_0, the unperturbed density.

We now look for a stationary wave propagating in the x direction, and take all physical quantities to be functions of $\xi = x - U_0 t$ only. Equations (2.1) then take the form $(\boldsymbol{v} = (u + U_0, 0\ 0))$

$$
\begin{aligned}
&nu = m = \text{const.}\\
&\mathrm{d}\phi/\mathrm{d}\xi = (u - u^{-1})\mathrm{d}u/\mathrm{d}\xi \qquad \text{(isothermal)}\\
&\qquad = (u - \gamma m^{\gamma-1}u^{-\gamma}\mathrm{d}u/\mathrm{d}\xi \qquad \text{(adiabatic)}\\
&\mathrm{d}^2\phi/\mathrm{d}\xi^2 = n - 1 = m/u - 1.
\end{aligned} \tag{2.2}
$$

These equations may be integrated once to give

$$
\begin{aligned}
\tfrac{1}{2}(\mathrm{d}u/\mathrm{d}\xi)^2 &= \frac{mu - \frac{1}{2}u^2 + mu^{-1} + \ln u + M}{(u - u^{-1})^2} \qquad \text{(isothermal)}\\
&= \frac{mu - \frac{1}{2}u^2 + m^{\gamma}u^{-\gamma} - \gamma m^{\gamma-1}u^{1-\gamma}/(\gamma-1) + P}{(u - \gamma m^{\gamma-1}u^{-\gamma})^2} \qquad \text{(adiabatic)}
\end{aligned} \tag{2.3}
$$

where M and P are constants.

In principle the explicit form of the waves could be obtained by integrating equation (2.3) over u, yielding a solution in the inverted form $\xi = \xi(u)$. However, to study the existence of solutions it is easier to consider the phase plane diagram where $\mathrm{d}u/\mathrm{d}\xi$ is considered as a function of u. This is of course given explicitly by equation (2.3), thus reducing the existence problem to one of plotting a curve in this phase space $(\mathrm{d}u/\mathrm{d}\xi, u)$. In this space, closed contours when they exist correspond to periodic nonlinear waves. For fixed values of m but differing values of M or P these contours are concentric about a centre.

In the present case a centre exists for $m > \gamma^{1/2}$ and thus this condition must necessarily be fulfilled for the existence of nonlinear waves (when not otherwise specified $\gamma = 1$ will describe the isothermal case from now on). In the immediate vicinity of the centre the contours are almost circular and correspond to small amplitude nonlinear waves, which are almost sinusoidal functions of ξ. As the amplitudes of these nonlinear waves increase, corresponding to changes in M or P, the contours in phase space steepen as they approach the singularity at $u = \gamma m^{\gamma-1}u^{-\gamma}$. The corresponding nonlinear waves take on a sawtooth-like profile. However there are no soliton-like solutions since no other critical point exists.

For $m < \gamma^{1/2}$, as stated above, no centre exists, so no closed contours exist and hence no stationary solutions which remain finite in amplitude for all ξ can exist. Yu (1976) claims to show the existence of soliton solutions for $m < \gamma^{1/2}$, but this conjecture is shown to be false in the Appendix.

In summary, nonlinear wave-like solutions exist, but soliton-type solutions do not.

Coffey (1971) has studied nonlinear plasma waves using the water-bag model. This is equivalent to the above if one takes $\gamma = 3$. By making an expansion in amplitude about the centre, Coffey obtains the explicit ξ dependence for weakly nonlinear waves. Ray (1978) has obtained somewhat equivalent results for isothermal plasmas.

By integrating the fourth equation of (2.2) over a period we obtain for the average value of u^{-1}

$$\langle u^{-1}\rangle = m^{-1} \tag{2.4}$$

for all wave solutions. Thus u must take values both below and above m, a detail that will be used in the following analysis. As the amplitude of the nonlinear wave tends to zero, the range of u values shrinks to a point at $u = m$.

3. The linear limit

We first solve for the zero-amplitude limit of the nonlinear wave, using equation (2.1) and writing

$$\begin{aligned} n &= 1 + \delta n \exp[\mathrm{i}(\boldsymbol{k}\,.\,\boldsymbol{x} - \omega t)] \\ \boldsymbol{v} &= m\boldsymbol{i}_x + \delta\boldsymbol{v} \exp[\mathrm{i}(\boldsymbol{k}\,.\,\boldsymbol{x} - \omega t)] \\ \phi &= \delta\phi \exp[\mathrm{i}(\boldsymbol{k}\,.\,\boldsymbol{x} - \omega t] \\ \boldsymbol{k}\,.\,\boldsymbol{i}_x &= k\cos\theta \end{aligned} \tag{3.1}$$

where $\boldsymbol{i}_x$ is a unit vector in the x direction.

Simple algebra yields

$$(\omega - km\cos\theta)^2 = 1 + \gamma k^2. \tag{3.2}$$

For the stationary wave $\omega = 0$, and if we consider $\theta = 0$ then

$$k_0^2 = (1 + \gamma k_0^2)/m^2 \tag{3.3}$$

so that

$$k_0^2 = (m^2 - \gamma)^{-1} > 0.$$

We now perturb this solution, introducing

$$\begin{aligned} \omega &= \delta\omega \\ \boldsymbol{k} &= \boldsymbol{k}_0 + \delta\boldsymbol{k} \\ \delta\boldsymbol{k}\,.\,\boldsymbol{i}_x &= \delta k\cos\theta' \end{aligned} \tag{3.4}$$

and this leads to

$$\delta\omega/\delta k = (m^2 - \gamma)/m\cos\theta' + O(\delta k). \tag{3.5}$$

Thus the problem is essentially two-dimensional. The phase velocity of the modulation $\delta\omega/\delta k$ is seen to be a circle in a polar plot of $V_{\mathrm{ph}}(\theta')$. This circle touches the origin on the right. It will be shown in the next section that, as we increase the amplitude of the basic stationary wave, this circle will bifurcate, giving rise to two circles that osculate at the origin. This bifurcation often accompanies nonlinear waves and has been observed in many different nonlinear problems (see, for example, Whitham 1965, 1974).

4. The dispersion relation for modulations

We now assume the existence of a nonlinear wave satisfying equation (2.2) and consider linear perturbations about this solution, which must of course satisfy equation (2.1). We assume that these linear equations have solutions of the form of a periodic function of ξ multiplied by $\exp[i(\boldsymbol{k}.\boldsymbol{x}-\omega t)]$ and the basic problem is to find the relation between ω and real k, that is the dispersion relation. The method to be used, which is basically an expansion in k, has been discussed in detail in relation to other nonlinear waves by Rowlands (1969, 1974) for one-dimensional perturbations, whilst three-dimensional ones have recently been discussed by Infeld *et al* (1978). Thus for example we write for the perturbed number density

$$\delta n(\boldsymbol{x}, t) = \delta n(\xi) \exp[i(\boldsymbol{k}.\boldsymbol{x}-\omega\tau)]$$

and assume that the $\delta n(\xi)$, which is a periodic function in ξ, may be expanded in powers of k such that $\delta n(\xi) = \delta n_0 + k\delta n_1 + \ldots$. We are using a double expansion: the first associated with the usual linearisation procedure is valid for $\delta n/n \ll 1$, whilst the second is used to facilitate the analytical solution of these linearised equations and is valid for $kL \ll 1$, where L is the spatial period of n. These two expansions are quite independent. Because of the translational invariance of equation (2.1) a solution to the linearised problem exists for $k=0$, where $\omega=0$ and $\delta n_0 = \mathrm{d}n/\mathrm{d}\xi$. Thus in general we write $\omega = \omega_1 k + \omega_2 k^2 + \ldots$. The perturbed velocity has a component δu in the direction of the nonlinear wave and one δv at right angles. In particular $\delta u_0 = \mathrm{d}u_0/\mathrm{d}\xi$ and $\delta v_0 = 0$. To first order in an expansion in k one obtains the set of equations

$$\begin{aligned}
&-\mathrm{i}\omega_1\delta n_0 + (\mathrm{d}/\mathrm{d}\xi)(n\delta u_1 + u\delta n_1) + \mathrm{i}\cos\theta(n\delta u_0 + u\delta n_0) = 0\\
&-\mathrm{i}\omega_1\delta u_0 + (\mathrm{d}/\mathrm{d}\xi)(u\delta u_1) + \mathrm{i}\cos\theta u\delta u_0 - (\mathrm{d}/\mathrm{d}\xi)(\delta\phi_1 - \gamma n^{\gamma-2}\delta n_1)\\
&\qquad -i\cos\theta(\delta\phi_0 - \gamma n^{\gamma-2}\delta n_0) = 0\\
&-\mathrm{i}\omega_1\delta u_0 + u(\mathrm{d}/\mathrm{d}\xi)\delta v_1 = \mathrm{i}\sin\theta(\delta\phi_0 - \gamma n^{\gamma-2}\delta n_0)\\
&\mathrm{d}^2\delta\phi_1/\mathrm{d}\xi^2 + 2\mathrm{i}\cos\theta(\mathrm{d}/\mathrm{d}\xi)\delta\phi_0 = \delta n_1
\end{aligned} \tag{4.1}$$

where we have written $\boldsymbol{k} = (k\cos\theta, k\sin\theta)$. Use has been made of the result $(-\mathrm{i}\omega_1 + \mathrm{i}ku\cos\theta)(n^{-\gamma}\delta p_0 - \gamma n^{-\gamma-1}p\delta n_0) + u(\mathrm{d}/\mathrm{d}\xi)(n^{-\gamma}\delta p_1 - \gamma n^{-\gamma-1}p\delta n_1) = 0$. It follows from the form of δn_0 and δp_0 that the first term is zero, so that $\delta p_1 = \gamma p n^{-1}\delta n_1$. The equations for the isothermal case can be obtained from the above by taking $\gamma = 1$, though this could not have been done in the initial equations because of the ln term.

Equation (4.1) constitutes four inhomogeneous equations for the unknown functions δ_1, δu_1, δv_1 and $\delta\phi_1$. Using the form for δn_0, δu_0, δv_0 and $\delta\phi_0$ mentioned above, it is readily seen that the first three equations of (4.1) are trivially integrated, and in particular δn_1 may be expressed algebraically in terms of $\delta\phi_1$ and known functions related to the nonlinear wave. When this form is substituted into the last equation of (4.1) one obtains a second-order inhomogeneous equation for $\delta\phi_1$. This equation may be solved by following the method given in the papers mentioned above. In this way one obtains the form for δn_1, δu_1, δv_1 and $\delta\phi_1$ but not the value of ω_1. To obtain this latter value one must proceed to second order in the expansion in k. Proceding in the manner outlined above, one obtains a second-order inhomogeneous differential equation for $\delta\phi_1$. Demanding that $\delta\phi_2$ must be periodic with the same periodicity as the nonlinear wave gives a consistency condition which involves $\omega_1(\sim\omega/k)$ and known functions and may be viewed as a dispersion relation. (To first order in the expansion in k the consistency condition is automatically satisfied, and for this reason we get no condition on ω_1.) In this way we obtain the result

$$\omega/k = \frac{m-\alpha_1/\alpha_0}{\eta_0-1}\cos\theta \pm \frac{\cos^2\theta+\eta\sin^2\theta}{(\eta_0-1)^2\alpha_0}\Bigg]^2 \times\left\{\left\langle\frac{(m/u-1)(1-\gamma(m/u)^\gamma/m^2)}{1-\gamma(m/u)^{\gamma+1}/m^2}\right\rangle\cos^2\theta + [m(\langle u\rangle - m)-J]\sin^2\theta(\eta_0-1)\right\}^{1/2} \tag{4.2}$$

where:

$\langle\ \rangle$ = average period of nonlinear wave;

α_n is defined by $\phi_\xi\int u^n\,\mathrm{d}\xi/2 = \alpha_n\xi\phi_\xi + \text{periodic } F(\xi)$;

$\eta_0 = \alpha_1^2/\alpha_0 - \alpha_2 - m^{-1}\langle(u^\xi - \gamma m^{\gamma-1}u^{-\gamma})^{-1}\rangle$;

$J = \langle\phi_\xi^2\rangle$.

The dispersion relation for the isothermal case may be obtained from equation (4.2) by taking $\gamma = 1$. For stationary waves with amplitudes tending to zero, equation (3.5) is recovered ($\omega/k \equiv \delta\omega/\delta k$). However, for general nonlinear waves there will be two dispersion waves in place of one (± in 4.2). Their topology will depend on whether the square root in equation (4.2) is positive or negative. This question will be examined in the next section.

5. Stability

To demonstrate stability we must show that $\omega(\boldsymbol{k})$ is real for all θ. This will be the case if and only if both the $\sin^2\theta$ and the $\cos^2\theta$ terms in the expression within the braces are non-negative. The proof of the first part of this condition is very simple. Write the coefficient of $\cos^2\theta$ before averaging as $(m/u-1)\pi(m,u)$. Then

$$\begin{aligned} \pi(m,u) &> 1 \text{ if } u<m \\ &= 1 \text{ if } u=m \\ &< 1 \text{ if } u>m. \end{aligned} \tag{5.1}$$

Therefore

$$\langle(m/u-1)\pi(m,u)\rangle > \langle(m/u-1)\rangle = 0 \tag{5.2}$$

(see ch 2).

The proof of the second part is slightly more involved. Denote

$$\begin{aligned} &s(u) = u - \gamma m^{\gamma-1}u^{-\gamma} > 0 \\ &s_u = \mathrm{d}s/\mathrm{d}u = 1+\gamma^2 m^{\gamma-1}u^{-\gamma-1} > 0 \\ &G = \mathrm{d}u/\mathrm{d}\xi, \qquad \lambda = \oint \mathrm{d}x \end{aligned} \tag{5.3}$$

and now proceed to evaluate $J+m(m-\langle u\rangle)$

$$\begin{aligned} J+m(m-\langle u\rangle) & \\ &=\langle s^2G^2\rangle+m\langle u\phi_{\xi\xi}\rangle=\langle s^2G^2\rangle+m\langle us_uG^2+usGG_u\rangle \\ &=\langle s^2G^2\rangle+m\langle us_uG^2\rangle+m/\lambda\oint usG_u/\mathrm{d}u \\ &=\langle s^2G^2\rangle+m\langle us_uG^2\rangle-m\langle (s+us_u)G^2\rangle \\ &=-\langle sG^2(m-s)\rangle=-sG^2(m-u)\rangle-\gamma m^{\gamma-1}\langle sG^2u^{-\gamma}\rangle. \end{aligned} \tag{5.4}$$

Now

$$\begin{aligned} -\langle sG^2(m-u)\rangle & \\ &=-\langle sG^2(us_uG^2+usGG_u)\rangle=-\langle sG^4us_u\rangle+\tfrac{1}{3}\langle G^4(2ss_uu+s^2)\rangle \\ &=-\tfrac{1}{3}\langle sG^4(us_u-s)\rangle=-\tfrac{1}{3}\langle sG^4(\gamma^2m^{\gamma-1}u^{-\gamma}+\gamma m^{\gamma-1}u^{-\gamma})\rangle<0. \end{aligned}$$

Now $\langle sG^2u^{-\gamma}\rangle>0$, and thus the LHS of equation (5.4) is negative. The proof is valid for $\gamma=1$ for the isothermal case. This then completes the proof of stability. Equation (4.2) has also been evaluated numerically and polar plots of ω/k as a function of θ obtained. When the amplitude of the nonlinear wave is zero this polar plot is in the form of a circle passing through the origin, as may be seen from equation (3.5). This solution is doubly degenerate since k_0, which must satisfy equation (3.3), can take either sign. As the amplitude of the basic nonlinear wave increases, this degeneracy is removed and the polar plot takes the form of two closed curves which closely resemble circles. These osculate at the origin but do not touch at any other point and thus for any θ there are two distinct real values of ω/k, showing that no unstable mode exists at least for small k.

For a cold plasma $\gamma=0$) the degeneracy of the two circles remains and one just recovers equation (3.5).

6. Conclusions

Conditions for the existence of electron plasma waves in isothermal and adiabatic plasmas have been given. Contrary to earlier work it has been shown that no soliton-type solution exists. The linear stability of these waves to long-wavelength three-dimensional perturbations has been demonstrated.

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Appendix

In this Appendix we take issue with some papers on the form of the nonlinear waves that can permeate an electron plasma. Yu (1976) claims that equation (2.3) can describe a *soliton* for values of u that include the singularity at

$$u=\gamma^{1/(\gamma+1)}m^{(\gamma-1)/(\gamma+1)}$$

if only $m<\gamma^{1/2}$. For simplicity we will treat the isothermal case in this refutation, so that the singularity is at $u=1$ and m must be smaller than 1.

One way of seeing that Yu must be wrong is to draw phase diagrams (u_ξ against u). It is then seen that the 'closed' phase curve in question is such that everywhere $u>m$. On the other hand, integration of the last of equations (2.2) over a large ξ domain which includes the soliton yields

$$(1/2L)\int_{-L}^{L} m/u\,\mathrm{d}\xi-1=(1/2L)\int_{-L}^{L}\mathrm{d}^2\phi/\mathrm{d}\xi^2\,\mathrm{d}\xi\sim 0$$

and this contradicts $u>m$. But to locate the mistake in his solution we must look for a less global argument.

In equation (9a) of the Yu reference, corresponding to our equation (2.3), different signs should be taken for $u<1$ and $u>1$ when the square root of both sides is taken. Thus, in our notation, assuming $u_\xi(1)>0$,

$$\begin{aligned}(u-u^{-1})\mathrm{d}u/\mathrm{d}\xi&=-\sqrt{f(u)} \qquad (u<1)\\ &=+\sqrt{f(u)} \qquad (u>1).\end{aligned}$$

The sign change will be from + to − if $u_\xi(1)<0$. As $f(1)$ is not zero, $(\mathrm{d}/\mathrm{d}\xi)(u-u^{-1})\mathrm{d}u/\mathrm{d}\xi=\mathrm{d}^2\phi/\mathrm{d}\xi^2$ will have a delta function contribution at $u=1$. On the other hand, $m/u-1$ is continuous there. Thus Yu's solution satisfies Poisson's equation

$$d^2\phi/d\xi^2=m/u-1$$

everywhere except at $u=1$. The same reasoning invalidates other papers by the same author (Yu 1977, 1978, Zhelyakor, *et al* 1978).

A third paper on the subject is by Shukla and Tagare (1977). In this paper, nonlinear waves such as are treated in the present paper are studied by making some quite unnecessary approximations. The authors state that solving their equation (8) is 'a formidable task' (this equation is the second-order differential equation for n obtainable from our equation (2.2)). In actual fact it is simply integrated by multiplying through by, in their notation,

$$-(\mathrm{d}/\mathrm{d}\xi)[U^2/2(1+N)^2+\ln(1+N)].$$

This enables us to write the whole equation as a perfect differential and, upon integration, leads to an equation similar to equation (2.3).

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